

A CHARACTERIZATION OF NILPOTENT LEIBNIZ ALGEBRAS

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ABSTRACT. W. A. Moens proved that a Lie algebra is nilpotent if and only if it admits an invertible Leibniz-derivation. In this paper we show that with the definition of Leibniz-derivation from [17] the similar result for non Lie Leibniz algebras is not true. Namely, we give an example of non nilpotent Leibniz algebra which admits an invertible Leibniz-derivation. In order to extend the results of paper [17] for Leibniz algebras we introduce a definition of Leibniz-derivation of Leibniz algebras which agrees with Leibniz-derivation of Lie algebras case. Further we prove that a Leibniz algebra is nilpotent if and only if it admits an invertible Leibniz-derivation. Moreover, the result that solvable radical of a Lie algebra is invariant with respect to a Leibniz-derivation was extended to the case of Leibniz algebras.

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1. INTRODUCTION

In 1955, Jacobson [11] proved that a Lie algebra over a field of characteristic zero admitting a non-singular (invertible) derivation is nilpotent. The problem, whether the inverse of this statement is correct, remained open until work [8], where an example of an nilpotent Lie algebra, whose derivations are nilpotent (and hence, singular), was constructed. Such types of Lie algebras are called characteristically nilpotent Lie algebras.

The study of derivations of Lie algebras lead to appearance of natural generalization – pre-derivations of Lie algebras [16]. In [2] it is proved that Jacobson’s result is also true in terms of pre-derivations. Similar to the example of Dixmier and Lister [8] several examples of nilpotent Lie algebras, whose pre-derivations are nilpotent were presented in [2], [4]. Such Lie algebras are called strongly nilpotent [4].

In paper [17] a generalized notion of derivations and pre-derivation of Lie algebras is defined as Leibniz-derivation of order k . In fact, a Leibniz-derivation is a derivation of a Leibniz k -algebra constructed by Lie algebra [6].

Below we present the characterization of nilpotency for Lie algebras in terms of Leibniz-derivations.

Theorem 1.1. [17] *A Lie algebra over a field of characteristic zero is nilpotent if and only if it has an invertible Leibniz-derivation.*

Leibniz algebras were introduced by Loday in [13]-[14] as a non-antisymmetric version of Lie algebras. Many results of Lie algebras are extended to Leibniz algebras case. Since the study of derivations and automorphisms of a Lie algebra plays essential role in the structure theory, the natural question arises whether the corresponding results for Lie algebras can be extended to more general objects.

In [12] it is proved that a finite dimensional complex Leibniz algebra admitting a non-singular derivation is nilpotent. Moreover, it was shown that similarly to the case of Lie algebras, the inverse of this statement does not hold and the notion of characteristically nilpotent Lie algebra can be extended for Leibniz algebras [18].

In this paper we show that if we define Leibniz-derivations for Leibniz algebra as in [17], then Theorem 1.1 does not hold. In order to avoid the confusion we need to modify the notion of Leibniz-derivation for Leibniz algebras.

Recall, in the definition of Leibniz-derivation of order k for Lie algebras the k -ary bracket is defined as multiplication of k elements on the left side. For the case of Leibniz algebras we propose the definition of Leibniz-derivation of order k as k -ary bracket on the right side. Due to anti-commutativity of multiplication in Lie algebras this definition agrees with the case of Lie algebras.

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Note that a vector space equipped with right sided k -ary multiplication is not a Leibniz k -algebra defined in [6]. For Leibniz-derivation of Leibniz algebra we prove the analogue of Theorem 1.1 for finite dimensional Leibniz algebras over a field of characteristic zero.

Through the paper all spaces and algebras are assumed finite dimensional.

2. PRELIMINARIES

In this section we present some known facts about Leibniz algebras and Leibniz n -algebras.

Definition 2.1. A vector space L over a field F with a binary operation $[-, -]$ is a (right) Leibniz algebra, if for any $x, y, z \in L$ the so-called Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds.

Every Lie algebra is a Leibniz algebra, but the bracket in a Leibniz algebra needs not to be skew-symmetric.

For a Leibniz algebra L consider the following central lower and derived sequences:

$$\begin{aligned} L^1 &= L, & L^{k+1} &= [L^k, L^1], & k &\geq 1, \\ L^{[1]} &= 1, & L^{[s+1]} &= [L^{[s]}, L^{[s]}], & s &\geq 1. \end{aligned}$$

Definition 2.2. A Leibniz algebra L is called nilpotent (respectively, solvable), if there exists $p \in \mathbb{N}$ ($q \in \mathbb{N}$) such that $L^p = 0$ (respectively, $L^{[q]} = 0$).

Levi's theorem, which has been proved for left Leibniz algebras in [3], is also true for right Leibniz algebras.

Theorem 2.3. (Levi's Theorem). Let L be a Leibniz algebra over a field of characteristic zero and R be its solvable radical. Then there exists a semisimple subalgebra Lie S of L , such that $L = S \dot{+} R$.

The following theorem from linear algebra characterizes the decomposition of a vector space into the direct sum of characteristic subspaces.

Theorem 2.4. [15] Let A be a linear transformation of the vector space V . Then V decomposes into the direct sum of characteristic subspaces $V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_k}$ with respect to A , where $V_{\lambda_i} = \{x \in V \mid (A - \lambda_i I)^k(x) = 0 \text{ for some } k \in \mathbb{N}\}$ and $\lambda_i, 1 \leq i \leq k$, are eigenvalues of A .

In Leibniz algebras a derivation is defined as follows

Definition 2.5. A linear transformation d of a Leibniz algebra L is a derivation if for any $x, y \in L$

$$d([x, y]) = [d(x), y] + [x, d(y)].$$

Consider for an arbitrary element $x \in L$ the operator of right multiplication $R_x : L \rightarrow L$, defined by $R_x(z) = [z, x]$. Operators of right multiplication are derivations of the Leibniz algebra L . The set $R(L) = \{R_x \mid x \in L\}$ is a Lie algebra with respect to the commutator, and the following identity holds:

$$R_x R_y - R_y R_x = R_{[y, x]}. \quad (2.1)$$

A subset S of an associative algebra A over a field F is called a *weakly closed subset* if for every pair $(a, b) \in S \times S$ there is an element $\gamma_{(a, b)} \in F$ such that $ab + \gamma_{(a, b)}ba \in S$.

We will need the following result concerning weakly closed sets

Theorem 2.6. [11] Let S be a weakly closed subset of the associative algebra A of linear transformations of a vector space V over F . Assume that every $W \in S$ is nilpotent, that is, $W^k = 0$ for some positive integer k . Then the enveloping associative algebra S^* of S is nilpotent.

The classical Engel's theorem for Lie algebras has the following analogue for Leibniz algebras.

Theorem 2.7. [1] A Leibniz algebra L is nilpotent if and only if R_x is nilpotent for any $x \in L$.

The following Theorem generalizes Jacobson's theorem to Leibniz algebras.

Theorem 2.8. [12] Let L be a complex Leibniz algebra which admits a non-singular derivation. Then L is nilpotent.

The next example presents n -dimensional Leibniz algebra possessing only nilpotent derivations.

Example 2.9. Let L be an n -dimensional Leibniz algebra and let $\{e_1, e_2, \dots, e_n\}$ be a basis of L with the following table of multiplication:

$$\begin{cases} [e_1, e_1] = e_3, \\ [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_1, e_2] = e_4, \\ [e_i, e_2] = e_{i+2}, & 2 \leq i \leq n-2, \end{cases}$$

(omitted products are equal to zero).

Using derivation property it is easy to see that every derivation of L has the following matrix form:

$$\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 & \dots & a_{n-1} & a_n \\ 0 & 0 & a_3 & a_4 & a_5 & \dots & a_{n-1} & b_n \\ 0 & 0 & 0 & a_3 & a_4 & \dots & a_{n-2} & a_{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & a_3 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Thus, every derivation of L is nilpotent, i.e., L is characteristically nilpotent.

Let us give the definition of Leibniz n -algebras.

Definition 2.10. [6] A vector space \mathcal{L} with an n -ary multiplication $[-, -, \dots, -] : \mathcal{L}^{\otimes n} \rightarrow \mathcal{L}$ is a Leibniz n -algebra if it satisfies the following identity:

$$[[x_1, x_2, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^n [x_1, \dots, x_{i-1}, [x_i, y_2, \dots, y_n], x_{i+1}, \dots, x_n]. \quad (2.2)$$

Let L be a Leibniz algebra with the product $[-, -]$. Then the vector space L can be equipped with a Leibniz n -algebra structure with the following product:

$$[x_1, x_2, \dots, x_n] = [x_1, [x_2, \dots, [x_{n-1}, x_n]]].$$

Definition 2.11. A derivation of a Leibniz n -algebra \mathcal{L} is a \mathbb{K} -linear map $d : \mathcal{L} \rightarrow \mathcal{L}$ satisfying

$$d([x_1, x_2, \dots, x_n]) = \sum_{i=1}^n [x_1, \dots, d(x_i), \dots, x_n].$$

The notion of Leibniz-derivation of Lie algebra was introduced in [17] and it generalizes the notions of derivation and pre-derivation of Lie algebra.

Definition 2.12. A Leibniz-derivation of order n for a Lie algebra G is an endomorphism P of G satisfying the identity

$$\begin{aligned} P([x_1, [x_2, \dots, [x_{n-1}, x_n]]]) &= [P(x_1), [x_2, \dots, [x_{n-1}, x_n]]] + \\ &+ [x_1, [P(x_2), \dots, [x_{n-1}, x_n]]] + \dots + [x_1, [x_2, \dots, [x_{n-1}, P(x_n)]]] \end{aligned}$$

for every $x_1, x_2, \dots, x_n \in G$.

In other words, a Leibniz-derivation of order n for a Lie algebra G is a derivation of G viewed as a Leibniz n -algebra.

3. LEIBNIZ-DERIVATION OF LEIBNIZ ALGEBRAS

The following example shows that Definition 2.12 is not substantial for the case of Leibniz algebras.

Example 3.1. Let R be an $(n+1)$ -dimensional solvable Leibniz algebra and $\{e_1, e_2, \dots, e_n, e_{n+1}\}$ be a basis of R with the table of multiplication given by

$$\begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_1, e_{n+1}] = e_2 + \sum_{i=4}^{n-1} \alpha_i e_i, & [e_2, e_{n+1}] = e_2 + \sum_{i=4}^{n-1} \alpha_i e_i, \\ [e_i, e_{n+1}] = e_i + \sum_{j=i+2}^n \alpha_{j-i+2} e_j, & 3 \leq i \leq n, \end{cases}$$

It is easy to see that $[R, [R, R]] = 0$. For the identity map d we have

$$0 = d([x, [y, z]]) = [d(x), [y, z]] + [x, [d(y), z]] + [x, [y, d(z)]] = 0.$$

Therefore, the invertible map d satisfies the condition of Definition 2.12, but the Leibniz algebra R is not nilpotent, i.e. analogue of Theorem 1.1 for Leibniz algebras is not true.

Remark 3.2. *The Example 3.1 can be extended for any non nilpotent solvable Leibniz algebra L such that L^2 lies in the right annihilator of L .*

Let us introduce n -ary multiplication as follows

$$[x_1, x_2, \dots, x_n]_r = [[[x_1, x_2], x_3] \dots, x_n].$$

The next example shows that, in general, a vector space equipped with defined n -ary multiplication $[x_1, x_2, \dots, x_n]_r$ is not a Leibniz n -algebra.

Example 3.3. *Let R be a solvable Leibniz algebra and let $\{e_1, e_2, \dots, e_n, x\}$ be a basis of R such that multiplication table of R in this basis has the following form [7]:*

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-1, \\ [x, e_1] = e_1, \\ [e_i, x] = -ie_i, & 1 \leq i \leq n. \end{cases}$$

It is not difficult to check that the vector space R with k -ary multiplication $[x_1, x_2, \dots, x_k]_r$ does not define Leibniz k -algebra structure. Indeed, we have

$$[[e_1, e_1, \dots, e_1]_r, x, \dots, x]_r = [\dots \underbrace{[[\dots [e_1, e_1], e_1], \dots, e_1]_r}_{k\text{-times}}, \underbrace{x, \dots, x}_{k-1\text{-times}}]_r = [\dots \underbrace{[e_k, x, x], \dots, x]_r}_{k-1\text{-times}}]_r = (-k)^{k-1} e_k.$$

On the other hand

$$\begin{aligned} \sum_{i=1}^k [e_1, \dots, e_1, \underbrace{[e_1, x, \dots, x]_r}_{i\text{-th}}, e_1, \dots, e_1]_r &= \sum_{i=1}^k [e_1, \dots, e_1, \underbrace{(-1)^{k-1} e_1}_{i\text{-th}}, e_1, \dots, e_1]_r = \\ &= (-1)^{k-1} \sum_{i=1}^k [\dots \underbrace{[e_1, e_1], e_1, \dots, e_1]_r}_{k\text{-times}}]_r = (-1)^{k-1} \sum_{i=1}^k e_k = (-1)^{k-1} k e_k. \end{aligned}$$

Hence identity (2.2) does not hold for $k \geq 3$.

Now we define the notion of Leibniz-derivation for Leibniz algebras.

Definition 3.4. *A Leibniz-derivation of order $n \in \mathbb{N}$ for a Leibniz algebra L is a \mathbb{K} -linear map $d : L \rightarrow L$ satisfying*

$$d([x_1, x_2, \dots, x_n]_r) = \sum_{i=1}^n [x_1, \dots, d(x_i), \dots, x_n]_r.$$

Proposition 3.5. *For Lie algebras Definition 3.4 agrees with Definition 2.12.*

Proof. Let L be a Lie algebra, then we have

$$P([x_1, [x_2, \dots, [x_{n-1}, x_n]]]) = (-1)^n P([[[x_n, x_{n-1}], \dots, x_2], x_1]) = (-1)^n P([x_n, x_{n-1}, \dots, x_1]_r).$$

On the other hand,

$$\begin{aligned} &[P(x_1), [x_2, \dots, [x_{n-1}, x_n]]] + [x_1, [P(x_2), \dots, [x_{n-1}, x_n]]] + \dots + [x_1, [x_2, \dots, [x_{n-1}, P(x_n)]]] = \\ &= (-1)^n [[[x_n, x_{n-1}], \dots, x_2], P(x_1)] + (-1)^n [[[x_n, x_{n-1}], \dots, P(x_2)], x_1] + \dots + (-1)^n [[[P(x_n), x_{n-1}], \dots, x_2], x_1] = \\ &= (-1)^n ([x_n, x_{n-1}, \dots, x_2, P(x_1)]_r + [x_n, x_{n-1}, \dots, P(x_2), x_1]_r + \dots + [P(x_n), x_{n-1}, \dots, x_2, x_1]_r) = \\ &= (-1)^n \sum_{i=1}^n [x_n, \dots, P(x_i), \dots, x_1]_r. \end{aligned}$$

This implies the equality

$$\begin{aligned} &P([x_1, [x_2, \dots, [x_{n-1}, x_n]]]) = [P(x_1), [x_2, \dots, [x_{n-1}, x_n]]] + \\ &+ [x_1, [P(x_2), \dots, [x_{n-1}, x_n]]] + \dots + [x_1, [x_2, \dots, [x_{n-1}, P(x_n)]]], \end{aligned}$$

which is equivalent to

$$P([x_n, x_{n-1}, \dots, x_1]_r) = \sum_{i=1}^n [x_n, \dots, P(x_i), \dots, x_1]_r.$$

Relabeling x_i with x_{n+1-i} for $1 \leq i \leq n$ we complete the proof of the proposition. \square

Let $LDer_n(L)$ denote the set of all Leibniz-derivations of order n for a Leibniz algebra L and let $LDer(L)$ be the set of all Leibniz-derivations, i.e. $LDer(L) = \bigcup_{n \in \mathbb{N}} LDer_n(L)$.

Note that a Leibniz derivation of order 2 is a derivation. Moreover, any derivation is a Leibniz-derivation of any order n . Thus, the order of a Leibniz-derivation is not unique.

Lemma 3.6. *The following statements are true*

- 1) If $s, t \in \mathbb{N}$ and $s|t$, then $LDer_{s+1}(L) \subset LDer_{t+1}(L)$;
- 2) for any $k, l \in \mathbb{N}$, $LDer_k(L) \cap LDer_l(L) \subset LDer_{k+l-1}(L)$.

Proof. The proof is similar to that of Lemma 2.3 [17]. \square

Similarly to the case of Lie algebras we call a Leibniz-derivation of order 3 a pre-derivation of Leibniz algebra. A nilpotent Leibniz algebra is called *strongly nilpotent* if all its Leibniz pre-derivations are nilpotent.

Note that a strongly nilpotent Leibniz algebra is characteristically nilpotent, but the inverse is not true in general.

Example 3.7. *Any pre-derivation of the characteristically nilpotent Leibniz algebra in Example 2.9 with $n = 6$ have the matrix form:*

$$\begin{pmatrix} a_1 & a_1 & a_3 & a_4 & a_5 & a_6 \\ 0 & 2a_1 & a_3 & a_4 & b_5 & b_6 \\ 0 & 0 & 3a_1 & -a_1 + a_3 & c_5 & c_6 \\ 0 & 0 & 0 & 4a_1 & 2a_1 + a_3 & a_4 \\ 0 & 0 & 0 & 0 & 5a_1 & a_1 + a_3 \\ 0 & 0 & 0 & 0 & 0 & 6a_1 \end{pmatrix}$$

Thus, this Leibniz algebra is not strongly nilpotent.

Proposition 3.8. *The Leibniz algebra L in Example 2.9 is strongly nilpotent if $n > 6$.*

Proof. Let $d : L \rightarrow L$ be a pre-derivation of L .

Put

$$d(e_1) = \sum_{i=1}^n a_i e_i, \quad d(e_2) = \sum_{i=1}^n b_i e_i, \quad d(e_3) = \sum_{i=1}^n c_i e_i.$$

Consider the property of pre-derivation

$$d(e_4) = d([e_1, e_1, e_1]_r) = (3a_1 + a_2)e_4 + (a_3 + 2a_2)e_5 + \sum_{i=4}^{n-2} a_i e_{i+2}.$$

On the other hand,

$$d(e_4) = d([e_2, e_1, e_1]_r) = (2a_1 + b_1 + b_2)e_4 + (b_3 + 2a_2)e_5 + \sum_{i=4}^{n-2} b_i e_{i+2}.$$

Comparing coefficients of basis elements we have

$$b_1 + b_2 = a_1 + a_2, \quad b_i = a_i, \quad 3 \leq i \leq n-2.$$

The equality $d([e_1, e_1, e_3]_r) = 0$ implies $0 = c_1 e_4 + c_2 e_5$, hence $c_1 = c_2 = 0$.

The chain of equalities

$$\begin{aligned} b_1 e_4 + (2a_1 + a_2 + b_2)e_5 + (a_3 + a_2)e_6 + \sum_{i=4}^{n-3} a_i e_{i+3} &= d([e_1, e_2, e_1]_r) = \\ d(e_5) = d([e_3, e_1, e_1]_r) &= (2a_1 + c_3)e_5 + (2a_2 + c_4)e_6 + \sum_{i=5}^{n-2} c_i e_{i+2}. \end{aligned}$$

deduce

$$b_1 = 0, \quad c_3 = a_2 + b_2, \quad c_4 = a_3 - a_2, \quad c_i = a_{i-1}, \quad 4 \leq i \leq n-2.$$

From the equalities

$$(3a_1 + 3a_2)e_6 + \sum_{i=3}^{n-4} a_i e_{i+4} = d([e_1, e_2, e_2]_r) = d(e_6) =$$

$$d([e_4, e_1, e_1]_r) = (5a_1 + a_2)e_6 + (4a_2 + a_3)e_7 + \sum_{i=4}^{n-4} a_i e_{i+4}$$

we get $a_1 = a_2 = 0$.

Since $b_2 = a_1 + a_2$ and $c_3 = a_2 + b_2$, we have $b_2 = c_3 = 0$.

Thus we obtain

$$d(e_1) = \sum_{i=3}^n a_i e_i, \quad d(e_2) = \sum_{i=3}^{n-2} a_i e_i + b_{n-1} e_{n-1} + b_n e_n, \quad d(e_3) = \sum_{i=3}^{n-3} a_i e_{i+1} + c_{n-1} e_{n-1} + c_n e_n.$$

Finally, from the expression $d([e_{i-2}, e_1, e_1]_r)$ we derive

$$d(e_i) = a_3 e_{i+1} + a_4 e_{i+2} + \cdots + a_{n+2-j} e_n, \quad i \geq 4$$

which completes the proof of Proposition. \square

Below we present 7- and 8-dimensional characteristically nilpotent Leibniz algebras, which are not strongly nilpotent.

Example 3.9. *The 7-dimensional Leibniz algebra with table of multiplication:*

$$\begin{cases} [e_1, e_1] = e_3, \\ [e_i, e_1] = e_{i+1}, & 2 \leq i \leq 6, \\ [e_1, e_2] = e_4 - 2e_5, \\ [e_i, e_2] = e_{i+2} - 2e_{i+3}, & 2 \leq i \leq 4, \\ [e_5, e_2] = e_7 \end{cases}$$

is characteristically nilpotent, but not strongly nilpotent.

Example 3.10. *The 8-dimensional filiform Leibniz algebra with table of multiplication:*

$$\begin{cases} [e_1, e_1] = e_3, \\ [e_i, e_1] = e_{i+1}, & 2 \leq i \leq 6, \\ [e_1, e_2] = e_4 - 2e_5 + 5e_6, \\ [e_i, e_2] = e_{i+2} - 2e_{i+3} + 5e_{i+4}, & 2 \leq i \leq 4, \\ [e_5, e_2] = e_7 - 2e_8 \\ [e_6, e_2] = e_8 \end{cases}$$

is characteristically nilpotent, but not strongly nilpotent.

Following the proofs of Lemmas in [9] and [5] for derivations of Lie and Leibniz n -algebras respectively, we get the following statement for Leibniz-derivations of order n of Leibniz algebras.

Lemma 3.11. *For a Leibniz-derivation $d : L \rightarrow L$ of order n of a Leibniz algebra L over a field of characteristic zero, the following formula holds for any $k \in \mathbb{N}$:*

$$d^k([x_1, \dots, x_n]_r) = \sum_{i_1+i_2+\dots+i_n=k} \frac{k!}{i_1!i_2!\dots i_n!} [d^{i_1}(x_1), d^{i_2}(x_2), \dots, d^{i_n}(x_n)]_r \quad (3.1).$$

4. NILPOTENT LEIBNIZ ALGEBRAS

Starting with a Leibniz algebra L , we denote the n -ary algebra with multiplication $[-, -, \dots, -]_r$ by $\mathcal{L}_n(L)$. A subalgebra I is called an n -ideal of L or an ideal of $\mathcal{L}_n(L)$, if it satisfies

$$\sum_{i=1}^n [L, \dots, I, \dots, L]_r \subseteq I.$$

Let M be any Leibniz subalgebra of L . Consider the following sequences:

$$\begin{aligned} \mathcal{L}_n^1(M) &= M, \quad \mathcal{L}_n^{k+1}(M) = [\mathcal{L}_n^k(M), M, \dots, M]_r, \quad k \geq 1, \\ \mathcal{L}_n^{[1]}(M) &= M, \quad \mathcal{L}_n^{[s+1]}(M) = [\mathcal{L}_n^{[s]}(M), \mathcal{L}_n^{[s]}(M) \dots, \mathcal{L}_n^{[s]}(M)]_r, \quad s \geq 1. \end{aligned}$$

Definition 4.1. A Leibniz algebra L is called n -nilpotent (n -solvable) if there exists a natural number $p \in \mathbb{N}$ ($q \in \mathbb{N}$) such that $\mathcal{L}_n^p(L) = 0$ ($\mathcal{L}_n^{[q]}(L) = 0$).

Lemma 4.2. Let M be an ideal of L . The following inclusions are true

$$\mathcal{L}_n^{[k]}(M) \subseteq M^{[k]}, \quad \mathcal{L}_n^k(M) \subseteq M^k.$$

Proof. It is easy to check that M^k and $M^{[k]}$ are also ideals of L for any k . We shall prove the first embedding by induction on k for any n .

If $k = 2$, then

$$\mathcal{L}_n^{[2]}(M) = [M, M, M, \dots, M]_r = [[[M, M], M], \dots, M] = [[M^{[2]}, M], \dots, M] \subseteq M^{[2]}.$$

Suppose that the statement holds for some k and we will prove it for $k + 1$.

$$\begin{aligned} \mathcal{L}_n^{[k+1]}(M) &= [[[\mathcal{L}_n^{[k]}(M), \mathcal{L}_n^{[k]}(M)], \mathcal{L}_n^{[k]}(M)], \dots, \mathcal{L}_n^{[k]}(M)] \subseteq \\ &\subseteq [[M^{[k]}, M^{[k]}], M^{[k]}], \dots, M^{[k]}] = [[M^{[k+1]}, M^{[k]}], \dots, M^{[k]}] \subseteq M^{[k+1]}. \end{aligned}$$

The second inclusion is established in a similar way. \square

Lemma 4.3. $M^{[t(k+1)]} \subseteq \mathcal{L}_n^{[k+1]}(M)$, where $k \in \mathbb{N}$ and t is a natural number such that $2^t \geq n$.

Proof. Since $M^{[p]} \subseteq M^{[p+q]}$ for any $p, q \in \mathbb{N}$, it is sufficient to prove embedding for the minimal t such that $2^t \geq n$.

We shall use induction. If $n = 3$ then $t = 2$.

For $k = 1$ we have

$$M^{[3]} = [M^{[2]}, M^{[2]}] = [M^{[2]}, [M^{[1]}, M^{[1]}]] \subseteq [[M^{[2]}, M^{[1]}], M^{[1]}] \subseteq [M^{[1]}, M^{[1]}, M^{[1]}]_r \subseteq \mathcal{L}_3^{[2]}(M).$$

Suppose that the statement holds for some k and we will prove it for $k + 1$.

$$\begin{aligned} M^{[2(k+1)+1]} &= M^{[2k+1+2]} = [[M^{[2k+1]}, M^{[2k+1]}], [M^{[2k+1]}, M^{[2k+1]}]] \subseteq \\ &\subseteq [[M^{[2k+1]}, M^{[2k+1]}], M^{[2k+1]}] \subseteq [\mathcal{L}_3^{[k+1]}(M), \mathcal{L}_3^{[k+1]}(M), \mathcal{L}_3^{[k+1]}(M)]_r = \mathcal{L}_3^{[k+2]}(M). \end{aligned}$$

Let us prove the statement for any n .

Since $2^t \geq n$ for $k = 1$ we get

$$M^{[t+1]} \subseteq M^{2^t} = \underbrace{[[M^{[1]}, M^{[1]}], \dots, M^{[1]}]}_{2^t - \text{times}} \subseteq \underbrace{[[M^{[1]}, M^{[1]}], \dots, M^{[1]}]}_{n - \text{times}} = \mathcal{L}_n^{[2]}(M).$$

The following chain equalities and inclusions

$$\begin{aligned} M^{[t(k+1)+1]} &= M^{[tk+1+t]} = (M^{[tk+1]})^{[t+1]} \subseteq (M^{[tk+1]})^{2^t} = \underbrace{[[M^{[tk+1]}, M^{[tk+1]}], \dots, M^{[tk+1]}]}_{2^t - \text{times}} \subseteq \\ &\subseteq \underbrace{[[M^{[tk+1]}, M^{[tk+1]}], \dots, M^{[tk+1]}]}_{n - \text{times}} \subseteq \underbrace{[\mathcal{L}_n^{[k+1]}(M), \mathcal{L}_n^{[k+1]}(M)], \dots, \mathcal{L}_n^{[k+1]}(M)]}_{n - \text{times}} = \mathcal{L}_n^{[k+2]}(M) \end{aligned}$$

complete the proof of the lemma. \square

Further we shall need the following lemma.

Lemma 4.4. $M^{nk-k+1} = \mathcal{L}_n^{k+1}(M)$.

Proof. The proof goes again by induction on k for any n .

If $k = 1$, then

$$M^n = \underbrace{[\dots [[M, M], M], \dots, M]}_{n\text{-times}} = [M, M, \dots, M]_r = \mathcal{L}_n^2(M).$$

Applying induction in the equalities

$$\begin{aligned} M^{n(k+1)-k-1+1} &= M^{nk-k+1+n-1} = [\dots [[M^{nk-k+1}, M], M], \dots, M] = \\ &= [M^{nk-k+1}, M, \dots, M]_r = [\mathcal{L}_n^{k+1}(M), M, \dots, M]_r = \mathcal{L}_n^{k+2}(M) \end{aligned}$$

we complete the proof of the lemma. \square

We denote by

R – solvable radical of L , i.e. the maximal solvable ideal of the Leibniz algebra L ;

R_n – n -solvable radical of L , i.e. the maximal n -solvable ideal of the n -ary algebra $\mathcal{L}_n(L)$;

N – nilradical of L , i.e. the maximal nilpotent ideal of the Leibniz algebra L ;

N_n – n -nilradical of L , i.e. the maximal n -nilpotent ideal of the n -ary algebra $\mathcal{L}_n(L)$.

Proposition 4.5. *For a Leibniz algebra L we have $R = R_n$.*

Proof. Lemma 4.2 implies that any solvable ideal of L is also n -solvable. Therefore, it is sufficient to prove the inclusion $R_n \subseteq R$. From Lemma 4.3 it follows that R_n is a solvable subalgebra of L . Thus, we need to prove that R_n is an ideal of L . According to Theorem 2.3, we can write $L = S \oplus R$, where S is a simple Lie algebra, R is a solvable ideal. Let $\pi : L \rightarrow S$ be the natural quotient map.

Since π is a morphism of L , we have

$$\begin{aligned} \pi([L, L, \dots, L, R_n]_r) &= \pi([[[[L, L], L], \dots, L], R_n]) = \\ &= [[[[\pi(L), \pi(L)], \pi(L)], \dots, \pi(L)], \pi(R_n)] = [[[[S, S], S], \dots, S], \pi(R_n)] = [S, \pi(R_n)]. \end{aligned}$$

On the other hand,

$$\pi([L, L, \dots, L, R_n]_r) \subseteq \pi(R_n).$$

Hence, $[S, \pi(R_n)] \subseteq \pi(R_n)$. Taking into account that S is a Lie algebra we obtain $[\pi(R_n), S] \subseteq \pi(R_n)$. Therefore, $\pi(R_n)$ is an ideal of S . Since R_n is an n -solvable ideal of L , $\pi(R_n)$ is an n -solvable ideal of S , consequently $\pi(R_n)$ is a solvable ideal (because $\pi(R_n)$ is an ideal of S).

Due to semisimplicity of S we get $\pi(R_n) = 0$, which implies $R_n \subseteq R$. \square

Lemma 4.6. *Let I be an ideal of the Leibniz algebra L and $d \in L\text{Der}_n(L)$ a Leibniz-derivation for some $n \in \mathbb{N}$. Then*

$$\mathcal{L}_n^{[k]}(d(I)) \subseteq I + d^{n^{k-1}}(\mathcal{L}_n^{[k]}(I))$$

for all $k \in \mathbb{N}$.

Proof. Evidently $d(I) \subseteq I + d(I)$ holds. For $k = 2$, using (3.1), we have

$$\begin{aligned} \mathcal{L}_n^{[2]}(d(I)) &= [d(I), d(I), \dots, d(I)]_r \subseteq d([I, I, \dots, I]_r) + \\ &+ \sum_{\substack{i_1 + i_2 + \dots + i_n = n \\ \exists i_j = 0}} \frac{n!}{i_1! i_2! \dots i_n!} [d^{i_1}(I), \dots, d^{i_{j-1}}(I), I, d^{i_{j+1}}(I), \dots, d^{i_n}(I)]_r \subseteq \\ &\subseteq I + d^n(\mathcal{L}_n^{[2]}(I)). \end{aligned}$$

Assume that $\mathcal{L}_n^{[k]}(d(I)) \subseteq I + d^{n^{k-1}}(\mathcal{L}_n^{[k]}(I))$. Again using (3.1), we verify the inclusion for $k + 1$:

$$\begin{aligned} \mathcal{L}_n^{[k+1]}(d(I)) &= [\mathcal{L}_n^{[k]}(d(I)), \mathcal{L}_n^{[k]}(d(I)), \dots, \mathcal{L}_n^{[k]}(d(I))]_r \subseteq \\ &\subseteq [I + d^{n^{k-1}}(\mathcal{L}_n^{[k]}(I)), I + d^{n^{k-1}}(\mathcal{L}_n^{[k]}(I)), \dots, I + d^{n^{k-1}}(\mathcal{L}_n^{[k]}(I))]_r \subseteq \\ &\subseteq I + d^{n^k}([\mathcal{L}_n^{[k]}(I), \mathcal{L}_n^{[k]}(I), \dots, \mathcal{L}_n^{[k]}(I)]_r) \subseteq I + d^{n^k}(\mathcal{L}_n^{[k+1]}(I)). \end{aligned}$$

\square

Theorem 4.7. *Let R be the solvable radical of a Leibniz algebra L over a field of characteristic zero. Then $d(R) \subseteq R$ for any $d \in L\text{Der}_n(L)$.*

Proof. Let d be a Leibniz-derivation of order n . Due to Proposition 4.5, $R = R_n$, so it is enough to prove the assertion of the Theorem for R_n .

Since R_n is a n -solvable radical, there exists $s \in \mathbb{N}$ such that $\mathcal{L}_n^{[s]}(R_n) = 0$. Then by Lemma 4.6, $\mathcal{L}_n^{[s]}(d(R_n)) \subseteq R_n + d^{n^{s-1}}(\mathcal{L}_n^{[s]}(R_n)) = R_n$. Thus, we have $\mathcal{L}_n^{[s]}(R_n + d(R_n)) \subseteq R_n$.

Further,

$$\mathcal{L}_n^{[2s-1]}(R_n + d(R_n)) \subseteq \mathcal{L}_n^{[s]}(\mathcal{L}_n^{[s]}(R_n + d(R_n))) \subseteq \mathcal{L}_n^{[s]}(R_n) = 0.$$

The n -ideal property of $R_n + d(R_n)$ follows from the following equalities:

$$\begin{aligned} [l_1, \dots, l_i + d(l_i), \dots, l_n]_r &= [l_1, \dots, l_i, \dots, l_n]_r + [l_1, \dots, d(l_i), \dots, l_n]_r = \\ &= [l_1, \dots, l_i, \dots, l_n]_r + d([l_1, \dots, l_i, \dots, l_n]_r) - \sum_{j=1, j \neq i}^n [l_1, \dots, d(l_j), \dots, l_n]_r. \end{aligned}$$

Hence $R_n + d(R_n)$ is an n -solvable ideal of the Leibniz algebra L . Since R_n is the n -solvable radical of L , it follows that $R_n + d(R_n) \subseteq R_n$, therefore $d(R_n) \subseteq R_n$. \square

Lemma 4.8. *Let I be an ideal of the Leibniz algebra L and $d \in L\text{Der}_n(L)$ a Leibniz-derivation for some $n \in \mathbb{N}$. Then*

$$\mathcal{L}_n^k(d(I)) \subseteq I + d^{kn-k+1}(\mathcal{L}_n^k(I))$$

for all $k \in \mathbb{N}$.

Proof. For $k = 1$ the assertion of the lemma is obvious. Let $k = 2$, then using the formula (3.1) we have

$$\begin{aligned} \mathcal{L}_n^2(d(I)) &= [d(I), d(I), \dots, d(I)]_r \subseteq d([I, I, \dots, I]_r) + \\ &+ \sum_{\substack{i_1 + i_2 + \dots + i_n = n \\ \exists i_j = 0}} \frac{n!}{i_1! i_2! \dots i_n!} [d^{i_1}(I), \dots, d^{i_j-1}(I), I, d^{i_j-1}(I), \dots, d^{i_n}(I)]_r \subseteq \\ &\subseteq I + d^n(\mathcal{L}_n^2(I)). \end{aligned}$$

Assume that $\mathcal{L}_n^k(d(I)) \subseteq I + d^{kn-k+1}(\mathcal{L}_n^k(I))$. Applying the formula (3.1), we prove the inclusion for $k + 1$:

$$\begin{aligned} \mathcal{L}_n^{k+1}(d(I)) &= [\mathcal{L}_n^k(d(I)), (d(I)), \dots, (d(I))]_r \subseteq [I + d^{kn-k+1}(\mathcal{L}_n^k(I)), d(I), \dots, d(I)]_r \subseteq \\ &\subseteq [I + d^{kn-k+1+n-1}([\mathcal{L}_n^k(I), I, \dots, I]_r) \subseteq I + d^{(k+1)n-k}(\mathcal{L}_n^{k+1}(I)). \end{aligned}$$

\square

Invariant property of nilradical of a Leibniz algebra under a Leibniz-derivation is presented in the following theorem.

Theorem 4.9. *Let N be the nilradical of a Leibniz algebra L over a field of characteristic zero. Then $d(N) \subseteq N$ for any $d \in L\text{Der}_n(L)$.*

Proof. The proof is similar to the proof of Theorem 4.7. \square

Next result establish properties of weight spaces with respect to a Leibniz-derivation of a Leibniz algebra.

Lemma 4.10. *Let L be a complex Leibniz algebra with a given Leibniz-derivation d of order n and $L = L_\alpha \oplus L_\beta \oplus \dots \oplus L_\gamma$ the decomposition of L into weight spaces with respect to d (i.e. $L_\alpha = \{x \in L : (d - \alpha 1)^k x = 0 \text{ for some } k\}$). Then*

$$[L_{\alpha_1}, L_{\alpha_2}, \dots, L_{\alpha_n}]_r = \begin{cases} 0 & \text{if } \alpha_1 + \alpha_2 + \dots + \alpha_n \text{ is not a root of } d \\ L_{\alpha_1 + \alpha_2 + \dots + \alpha_n} & \text{if } \alpha_1 + \alpha_2 + \dots + \alpha_n \text{ is a root of } d. \end{cases}$$

Proof. First observe that

$$\begin{aligned} (d - (\alpha_1 + \alpha_2 + \dots + \alpha_n) \cdot 1)[x_1, x_2, \dots, x_n]_r &= \\ \sum_{i=1}^n [x_1, \dots, d(x_i), \dots, x_n]_r - \sum_{i=1}^n [x_1, \dots, \alpha_i x_i, \dots, x_n]_r &= \sum_{i=1}^n [x_1, \dots, (d - \alpha_i \cdot 1)x_i, \dots, x_n]_r. \end{aligned}$$

Similarly to Lemma 3.11, by induction on k we get the following equality:

$$(d - (\alpha_1 + \alpha_2 + \dots + \alpha_n) \cdot 1)^k [x_1, x_2, \dots, x_n]_r =$$

$$= \sum_{i_1+i_2+\dots+i_n=k} \frac{k!}{i_1!i_2!\dots i_n!} [(d - \alpha_1 \cdot 1)^{i_1} x_1, (d - \alpha_2 \cdot 1)^{i_2} x_2, \dots, (d - \alpha_n \cdot 1)^{i_n} x_n]_r \quad (4.1)$$

for any $x_i \in L_{\alpha_i}$.

Consider $x_i \in L_{\alpha_i}$, $1 \leq i \leq n$. Then there exist natural numbers k_i such that $(d - \alpha_i \cdot 1)^{k_i}(x) = 0$. Taking $k = \sum_{i=1}^n k_i$ in (4.1), we have

$$(d - (\alpha_1 + \alpha_2 + \dots + \alpha_n) \cdot 1)^k [x_1, x_2, \dots, x_n]_r = 0$$

which completes the proof. \square

Similarly as in [17] we have the existence of an invertible Leibniz-derivation of nilpotent Leibniz algebra.

Proposition 4.11. *Every nilpotent Leibniz algebra with nilindex equal to s has an invertible Leibniz-derivation of order $\lceil \frac{s}{2} \rceil + 1$.*

Proof. Let L be a Leibniz algebra with nilindex equal to s and set $q = \lceil \frac{s}{2} \rceil + 1$. Consider the vector subspace W of L complementary to L^q , i.e. $L = W + L^q$. Define the map P by the following way:

$$P(x) = \begin{cases} x & \text{if } x \in W, \\ qx & \text{if } x \in L^q. \end{cases}$$

It is easy to check that P is a Leibniz-derivation for L of order q . \square

Below we present one of the main theorems of the paper.

Theorem 4.12. *Let L be a complex Leibniz algebra which admits an invertible Leibniz-derivation. Then L is nilpotent.*

Proof. Let d be an invertible Leibniz-derivation of order n of the Leibniz algebra L and

$$L = L_{\rho_1} \oplus L_{\rho_2} \oplus \dots \oplus L_{\rho_s}$$

be the decomposition of L into characteristic spaces with respect to d .

Let $\alpha, \beta \in \text{spec}(d)$. Then by Lemma 4.10 we have

$$[L_\alpha, \underbrace{L_\beta, L_\beta, \dots, L_\beta}_{n-1\text{-times}}]_r = [\dots [L_\alpha, \underbrace{L_\beta, L_\beta, \dots, L_\beta}_{n-1\text{-times}}] \dots] \subseteq L_{\alpha+(n-1)\beta}.$$

Considering k -times of the n -ary multiplication we have

$$\begin{aligned} & [\dots [L_\alpha, \underbrace{L_\beta, L_\beta, \dots, L_\beta}_{n-1\text{-times}}]_r, \underbrace{L_\beta, L_\beta, \dots, L_\beta}_{n-1\text{-times}}]_r, \dots, \underbrace{L_\beta, L_\beta, \dots, L_\beta}_{n-1\text{-times}}]_r = \\ & \underbrace{\dots [L_\alpha, \underbrace{L_\beta, L_\beta, \dots, L_\beta}_{n-1\text{-times}}]_r, \underbrace{L_\beta, L_\beta, \dots, L_\beta}_{n-1\text{-times}}]_r, \dots, \underbrace{L_\beta, L_\beta, \dots, L_\beta}_{n-1\text{-times}}]_r}_{k\text{-times}} = \\ & [\dots [L_\alpha, \underbrace{L_\beta, L_\beta, \dots, L_\beta}_{k(n-1)\text{-times}}] \dots] \subseteq L_{\alpha+k(n-1)\beta}. \end{aligned}$$

Since for sufficiently large $k \in \mathbb{N}$ we obtain $\alpha + k(n-1)\beta \notin \text{spec}(d)$, by Lemma 4.10 we obtain $[\dots [L_\alpha, \underbrace{L_\beta, L_\beta, \dots, L_\beta}_{k(n-1)\text{-times}}] \dots] = 0$.

Thus, any operator of right multiplication $R_x : L \rightarrow L$, where $x \in L_\beta$, is nilpotent and, due to the fact that α, β were taken arbitrary, it follows that every operator from $\bigcup_{i=1}^k R(L_{\rho_i})$ is nilpotent.

Now from identity (2.1) and Lemma 4.10 it follows that $\bigcup_{i=1}^k R(L_{\rho_i})$ is a weekly closed set of an associative algebra $\text{End}(L)$. Hence, by Theorem 2.6 it follows that every operator from $R(L)$ is nilpotent.

Hence, R_x is nilpotent for any $x \in L$. Now by Engel's Theorem (Theorem 2.7) we conclude that L is nilpotent. \square

Finally from the Theorem 4.12 and Proposition 4.11 we get the analogue of Theorem 1.1 for Leibniz algebras.

Theorem 4.13. *A Leibniz algebra over a field of characteristic zero is nilpotent if and only if it has an invertible Leibniz-derivation.*

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